Introduction to Graph Theory Notes (2023/2024)

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Contents

A **graph** is a pair $G = (V, E)$ where V is the set of points, called **vertices** and $E \subseteq \binom{V}{2}$ is a set of pairs of vertices, called **edges**. (Here we use the notation $\binom{V}{2} := \{\{u, v\} : u \neq v \in V\}$.
For notational convenience we usually write $uv \in E$ in place of $\{u, v\} \in E$.
Sometimes the notation $u \sim v$ is also used for $uv \in E$.

If G is a graph then we use the notations $V(G), E(G)$ for the vertex set and edge set of G, and we denote by $v(G) := |V(G)|, e(G) := |E(G)|$, the number of vertices, respectively edges of G

We say a vertex v is **incident** with an edge $e \in E$ if v is one of the endpoints of e.
Two vertices $u, v \in V(G)$ are **neighbors** if $uv \in E(G)$. We denote the set of neighbors of v by $N(v)$.
The **degree** of v is its number of neighbours: $\deg(v) := |N(v)|$.
A graph H is a **subgraph** of another graph G, denoted $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

1 Walks and cycles

A **walk** in G with length k is a sequence of vertices $v_0, v_1, \ldots, v_k \in V(G)$ such that $v_i v_{i+1} \in E(G)$ for $i = 0, \ldots, k-1$. (Note: repetitions of vertices is allowed.)
The **length** of the walk is the number of edges that is traversed by the walk, including multiplicities.

A **closed walk** is a walk where the starting point and end point coincide.
A **path** is a walk where all vertices are distinct.
A **cycle** is a closed walk where all vertices are distinct, and $k \geq 3$.
A (closed) walk and a path can have length 0, but a cycle is always required to have positive length.

1.1 Connected and disconnected graphs

A graph G is **connected** if, for every $u \neq v \in V(G)$ there exists a path between u and v.
It is **disconnected** if it is not connected.
A graph is **complete** (or a **clique**) if $uv \in E(G)$ for all $u \neq v \in V(G)$.

Any graph naturally decomposes into groups of connected vertices, called (connected) **components**. A connected component H of the graph G is a **maximal** connected subgraph, where maximal means that we cannot add any vertices and edges of G that are not already in H without creating a disconnected graph. A graph is connected if and only if it has precisely one component.

Lemma If G is connected and contains a cycle C then $G \setminus e$ is connected for every $e \in E(C)$.

Lemma If G is finite and $\deg(v) \geq 2$ for every vertex of G, then G has a cycle.

If $e(G) \geq v(G)$, then G has a cycle.

1.2 Euler tours

We call a closed walk that uses every edge exactly once and visits every vertex of a graph an **Euler circuit**, or **Euler tour**. The graph G is said to be **Eulerian** if it admits such an Euler tour.

Theorem [Euler] A finite graph is Eulerian if and only if it is connected and all degrees are even.

1.3 Hamilton cycles

$G \setminus v$ denotes the graph obtained by removing v and all incident edges from G.
A **cut vertex** is a vertex v such that $G \setminus v$ is disconnected.

We call a closed walk that visits every vertex exactly once a **Hamilton cycle**.
G cannot have a Hamilton cycle if there is a cut vertex.

Theorem If G has a Hamilton cycle then $G \setminus S$ has at most $|S|$ components for every non-empty $S \subseteq V(G)$.

Theorem [Dirac] Let G be a graph with at least three vertices, satisfying $\deg(v) \geq \frac{v(G)}{2}$, for all $v \in V(G)$.
Then G has a Hamilton cycle.

2 Binary encoding

Binary expansion of an integer: $n = \sum_{i=0}^{\infty} b_i * 2^i$

A **Gray code** is an encoding scheme where the codes for i and $i+1$ differ in only one bit.

We denote by Q_k the k-dimensional binary hypercube.
That is the graph with vertex set $V(Q_k) = \{0,1\}^k$ (all bit strings of length k)
and edge set $E(Q_k) = \Big\{xy \in \binom{\{0,1\}^k}{2} : ||x-y|| = 1\Big\}$
(edges between bitstrings that differ in precisely one coordinate)
For all $k \geq 2$, the graph Q_k has a Hamilton cycle.

2.1 De Bruijn sequences

Fix $n \geq k \geq 2$ and A a finite set of cardinality at least two (the **alphabet**).
We say a vector $(y_1, \ldots, y_k) \in A_k$ occurs as a consecutive subsequence in $(x_0, \ldots, x_{n-1}) \in A_n$ if there is an $0 \leq i \leq n-1$ such that $x_i = y_1, \quad x_{(i+1)} \bmod n = y_2, \quad \ldots \quad , x_{(i+k-1) \bmod n} = y_k$
When every subsequence of length k appears exactly once, it is called a **De Bruijn sequence**.

Theorem [Van Aardenne-Ehrenfest, De Bruijn]

For every $k \geq 1$ and alphabet size $r \geq 2$, there exists a (circular) sequence that contains each of the r^k sequences of length k precisely once as consecutive subsequences.

2.2 De Bruijn graphs

A **directed graph** has **arcs**, which are directed edges.
A directed graph D is Eulerian if and only if:

1. For every $u, v \in V$ there is a directed path from u to v (D is **strongly connected**)
2. For every $v \in V$, the number of arcs coming into v equals the number of arcs going out of v

A De Bruijn sequence can be represented by a **De Bruijn graph**. This is a directed Eulerian graph, whose vertices are all the sequences $(w_1, \ldots, w_{k-1}) \in A_{k-1}$ of length $k-1$.
There is an arc from $(w_1, \ldots, w_{k-1})$ to $(v_1, \ldots, v_{k-1})$ if and only if $v_1 = w_2, v_2 = w_3, ..., v_{k-2} = w_{k-1}$.
We label this arc with $(w_1, \ldots w_{k-1}, v_{k-1})$, a **word** of length k.

3 Trees

A **tree** is a graph that is connected and acyclic. A **leaf** is a vertex of degree 1.

Lemma A tree with at least 2 vertices has at least 2 leaves.

Lemma If a leaf is removed from a tree, it will still be a tree.

A tree T is a **spanning tree** of G if $V(G) = V(T)$.

Corollary Any connected graph contains a spanning tree.

Theorem The following are equivalent:

1. T is a tree
2. T is connected and $v(T) - 1 = e(T)$
3. T is **minimally connected**: T is connected, but $T \setminus e$ is disconnected $\forall e \in E(T)$
4. T is acyclic and $v(T) - 1 = e(T)$
5. T is **maximally acyclic**: T is acyclic, but $T \cup e$ has a cycle $\forall e \in \binom{V(T)}{2} \setminus E(T)$
6. $\forall u, v \in V(T)$ there is exactly 1 path between u and v

4 Weighted graphs

4.1 Minimum spanning tree

A graph can be given **edge weights**, defined by a function $w : E \to [0, \infty)$
A **minimum spanning tree** (MST) is a spanning tree of G such that the sum of its edge-weights is as small as possible subject to being a spanning tree of G.

Kruskal's algorithm: Starting from the lowest weight, repeat for every edge:
If adding the edge does not create a cycle, add the edge.

Theorem [Kruskal] Kruskal's algorithm always produces a minimum spanning tree.

Two graphs **agree** on an edge if both graphs either contain the edge or do not contain it.
They **disagree** on an edge if only one of the graphs contains the edge.

The **Dijkstra-Jarnik-Prim algorithm** is a variation on Kruskal's algorithm where we start with the cheapest edge and repeatedly add the cheapest edge that is incident with the subgraph and does not create a cycle. This algorithm also always computes an MST.

4.2 Shortest paths

Dijkstra's Algorithm for computing the shortest path:
S is the set of vertices of which the distance to u is known, $t(v)$ is the tentative distance from u to some vertex v, and $w(xy)$ is the weight of an edge xy, where $w(xy) = \infty$ if it is not an edge.

- Initialization: Set $S = \{u\}, \quad t(u) = 0, \quad t(z) = w(uz) \;\; \forall z \in V$
- Iteration:
  - Select a vertex $v \notin S$ with minimal $t(v)$. Add v to S.
  - For each edge vz with $z \notin S$, update $t(z)$ to $\min\{t(z),\, t(v) + w(vz)\}$.
  - Repeat until $S = V(G)$.
- At the end, set $d(u, v) = t(v)$ for all $v \in V$.

Theorem [Dijkstra] Dijkstra's Algorithm correctly computes $d(u, z)$ for all $z \in V$

5 Matchings

A **matching** M is a subset of the edges of a graph, where the edges do not share endpoints.
M is **perfect** if every vertex is **saturated** by M: every vertex is an endpoint of some edge in M.
We say that a matching M is **maximum** if there is no matching of larger cardinality.
In contrast, we say M is **maximal** if it is not possible to add another edge $e \in E \setminus M$ without violating the condition of being a matching.

A path P is M**-alternating** if either the odd-numbered or even-numbered edges of P are all in M.
The path P is M**-augmenting** if P is M-alternating and both endpoints of P are not saturated.

Theorem [Berge] A matching M is maximum if and only if there is no M-augmenting path.

Lemma Every component of the symmetric difference of two matchings is either a path or an even cycle.

5.1 Partitions

A graph $G = (V, E)$ is called **bipartite** if we can partition $V = X \uplus Y$ into two parts such that every edge has one endpoint in X and one endpoint in Y.
More generally, G is k**-partite** if $V = V_1 \uplus \ldots \uplus V_k$ can be partitioned into k parts such that, for each $i = 1, \ldots, k$, there is no edge with both endpoints in V_i.

Theorem
[Hall's marriage theorem]

> Let $G = (V, E)$ be a bipartite graph with bipartition $V = X \uplus Y$.
> There is a matching saturating X if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$, where $N(S)$ is the set of neighbors of vertices in S.

We say a graph G is **k-regular** if $\deg(v) = k$ for all $v \in V(G)$.

Corollary

> If G is bipartite and k-regular (with $k \geq 1$) then G has a perfect matching.

5.2 Covers

$C \subseteq V(G)$ is a **cover** of G if every edge has at least 1 endpoint in C.

For all matchings $M \subseteq E(G)$ and covers $C \subseteq V(G)$, $|M| \leq |C|$.
If $|M| = |C|$, then M is maximum and C is minimum.

Theorem
[König, Egerváry]

> If G is bipartite, $\max\limits_{M \text{ matching}} |M| = \min\limits_{C \text{ cover}} |C|$.

5.3 Bipartite matchings with preferences

If x is matched with y' and y is matched with x', but x prefers y over y' and y prefers x over x', then the unmatched pair (x, y) is an **unstable pair**.
A perfect matching is a **stable matching** if it yields no unmatched unstable pairs.

Gale-Shapley Proposal Algorithm
Let X be a set of girls and Y a set of boys where $|X| = |Y|$.

1. Every boy proposes to the girl highest on his preference list which has not previously rejected him.
2. If each girl receives exactly 1 proposal, stop and use the resulting matching.
3. Otherwise, every girl receiving more than one proposal rejects all of them except the one highest on her preference list.
4. Every girl says "maybe" to the most attractive proposal.

Theorem
[Gale, Shapley]

> The Gale-Shapley algorithm always produces a stable matching.

5.4 Maximum and perfect matchings

A graph can only have a perfect matching if $V(G)$ is even.
Notation: $o(G) := \#\{\text{components of G with an odd number of vertices}\}$

Theorem
[Tutte]

> A graph G has a perfect matching if and only if $o(G \setminus S) \leq |S|$ for every $S \subseteq V(G)$.

Corollary
[Berge-Tutte Formula]

$$\max_{M \text{ matching}} |M| = \frac{1}{2}\left(\min_{S \subseteq V(G)} v(G) - o(G \setminus S) + |S|\right)$$

Guessing some S will lead to an upper bound for $|M|$.
Guessing some M will lead to a lower bound for $|M|$.

6 Planar graphs

A graph is called **planar** if it can be drawn in the plane without intersecting edges.

A **drawing** assigns a point $p(v) \in \mathbb{R}^2$ to each vertex $v \in V(G)$
and a **curve** $c(e)$ (a continuous map $[0, 1] \to \mathbb{R}^2$) with endpoints $p(u), p(v)$ to each edge $uv \in E(G)$.
A planar graph has such a drawing in which c(e), c(f) do not intersect, except at common endpoints.

We say a curve is **closed** if it is the image of a continous map $\varphi : [0, 1] \to \mathbb{R}^2$ with $\varphi(0) = \varphi(1)$.
A curve is closed and **simple** if $\varphi(0) = \varphi(1)$ is the only repeated value. (no self-intersections)

A set $A \subseteq R^2$ is **path-connected** if for every $a, b \in A$ there is a curve $c \subseteq A$ with endpoints a, b.
A **path-connected component** of a subset $A \subseteq R^2$ of the plane is defined analogously to a connected component: it is a maximal path-connected subset of A.

Theorem [Jordan curve theorem]

If c is a simple closed curve then $R^2 \setminus c$ consists of precisely two path-connected components.

A **plane graph** is a planar graph together with a fixed drawing of that graph.
A cycle in a plane graph is a simple closed curve.
The union of the curves and points in a plane graph partitions the rest of the plane R^2 into one or more path-connected components, which we call **faces**.

A k**-face** is a face with k edges on its boundary. Here an edge can occur more than once.
There will always be one unbounded face, which we sometimes refer to as the **infinite face**.

K_n denotes the complete graph on n vertices.
$K_{n,m}$ denotes the complete bipartite graph with $|X| = n, |Y| = m$.

Theorem [Kuratowski] A graph is planar if and only if it contains neither a subdivision of $K_{3,3}$ nor a subdivision of K_5 as a subgraph.

Theorem [Euler's formula] For every connected plane graph we have $v(G) - e(G) + f(G) = 2$

Lemma If G is planar then $e(G) \leq 3v(G)$. If G is planar and $v(G) \geq 3$ then $e(G) \leq 3v(G) - 6$.

Corollary Every planar graph has a vertex of degree at most five.

7 Colouring

A k**-colouring** of a graph G is a map $\varphi : V(G) \to [k]$ such that $\varphi(u) \neq \varphi(v)$ whenever $uv \in E(G)$.
The **chromatic number** is $\chi(G) = \min\{k : G \text{ has a } k\text{-colouring}\}$.

Corollary [Six colour theorem] $\chi(G) \leq 6$ for every planar graph G.

Theorem [Heawood's five colour theorem] $\chi(G) \leq 5$ for every planar graph G.

$\Delta(G)$ is the maximum vertex degree in G, $\delta(G)$ is the minimum vertex degree in G.
The **clique number** $\omega(G)$ is the number of vertices in the largest clique of G.

Lemma $\chi(G) \leq \Delta(G) + 1$ for all graphs G.

Lemma $\chi(G) \geq \omega(G)$ for all graphs G.

A **stable set** or **anti-clique** is a set $A \subseteq V(G)$ such that there are no edges between vertices in A.
The **stability** $\alpha(G)$ is the cardinality of the largest stable set in G.

Lemma $\chi(G) \geq \dfrac{v(G)}{\alpha(G)}$ for all graphs G.

If G is a graph with vertex set $V = \{v_1, \ldots, v_n\}$ then the **Mycielskian** $M(G)$ of G is the graph with vertex set $V \cup \{w_1, \ldots, w_n\} \cup \{z\}$ and edge set
$E(G) = \{\text{the edges of the original } G\} \cup \{v_i w_j : v_i v_j \text{ is an edge of } G\} \cup \{w_i z : i = 1, \ldots, n\}$
G is triangle-free $\implies M(G)$ is triangle-free

Theorem [Mycielski] $\chi(M(G)) = \chi(G) + 1$.

The **girth** of a graph G is the length of the shortest cycle in G.

Theorem [Erdös] For every k, l there exists a graph G with $\chi(G) > k$ and $\text{girth}(G) > l$.

7.1 List colouring

A **list colouring** of G is a map $\phi : V(G) \to \mathbb{N}$ such that $\phi(v)$ is in the list $L(v)$.
List chromatic number:
$\chi_l(G) := \min\{k : \text{every list assignment with } |L(v)| > k \text{ for all } v \in V(G) \text{ has an } L\text{-colouring}\}$

$$\chi_l(G) \geq \chi(G) \qquad \chi_l(G) \leq \Delta(G) + 1$$

Theorem [Erdös, Rubin, Taylor] For every k, there exists a bipartite graph with $\chi_l(G) > k$.

Theorem [Thomassen] $\chi_l(G) \leq 5$ for every planar graph G.

A plane graph is a **near triangulation** if every face is a 3-face except possibly the outer face.

7.2 Edge colouring

When colouring edges, edges that share endpoints have to be different colours.
Notation for edge colouring: $\varphi : E(G) \to [k]$
A **line graph** $L(G)$ has vertex set $V(L(G)) := E(G)$
and edge set $E(L(G)) := \{ef : e, f \in E(G) \text{ share an endpoint}\}$
The **edge chromatic number** $\chi'(G)$ is the chromatic number of the line graph.
$\omega(L(G)) = \Delta(G)$ unless $\Delta(G) = 2$ and G contains a triangle, in which case $\omega(L(G)) = 3$

$$\Delta(G) \leq \omega(L(G)) \leq \chi'(G) \leq \Delta(L(G)) + 1 \leq 2\Delta(G) - 1$$

Theorem [König] If G is bipartite then $\chi'(G) = \Delta(G)$

For every bipartite graph G there exists a bipartite and $\Delta(G)$-regular graph H such that $G \subseteq H$

Theorem [Vizing] $\chi'(G) \leq \Delta(G) + 1$

A **class one graph** has $\chi'(G) = \Delta(G)$ and a **class two graph** has $\chi'(G) = \Delta(G) + 1$.

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1 \qquad \Delta(G) \leq \chi'_l(G) \leq \Delta(G) + 1$$

8 Connectivity

A **cut set** or **separating set** is a subset $S \subseteq V(G)$ such that $G \setminus S$ is disconnected.
G is **k-connected** if $v(G) > k$ and for every set $S \subseteq V(G)$ of cardinality $|S| < k$ we have that $G \setminus S$ is connected. The **connectivity** κ of G is the largest k such that G is k-connected.

An **internal vertex** of a path is a vertex on the path that is not an endpoint.
We say that paths are **internally vertex disjoint** if no pair of them shares an internal vertex.

Theorem [Menger] A graph is k-connected if and only if between any two distinct vertices there are at least k internally vertex disjoint paths.

For $A, B \subseteq V(G)$ (not necessarily disjoint), an (A, B)**-path** is a path with one endpoint in A, one endpoint in B and all internal vertices (if there are any) in $V(G) \setminus (A \cup B)$.
We'll say that $S \subseteq V(G)$ is an (A, B)**-separator** if $G \setminus S$ does not contain any (A, B)-path.
An (A, B)**-connector** is a set of (A, B)-paths that are (vertex) disjoint, with distinct endpoints.

Theorem [Pym] The minimum size of an (A, B)-separator is the maximum size of an (A, B)-connector.

8.1 Edge connectivity

A set of edges $F \subseteq E(G)$ is called an **edge separator** if $G \setminus F$ is disconnected.
G is k**-edge connected** if every edge separator has cardinality $\geq k$.
The **edge connectivity** $\kappa'(G)$ is the largest k such that G is k-edge connected.

Theorem [Whitney] $\kappa(G) \leq \kappa'(G) \leq \delta(G)$

Theorem [Menger's theorem for edge connectivity]

A graph is k-edge connected iff between any pair of vertices there are $\leq k$ edge disjoint paths.

8.2 Ears

An **ear** in a graph is a path where internal vertices have degree two and endpoints have degree ≥ 3.

An **ear decomposition** of a graph G is a sequence $G_0 \subseteq \ldots \subseteq G_k$ such that G_0 is a cycle, $G_k = G$ and G_i is obtained from G_{i-1} by adding an ear.

Theorem [Whitney] A graph is 2-connected if and only if it has an ear decomposition.

9 Extremal graph theory

An **extremum** is a maximum or a minimum.

Theorem [Mantel's theorem] Suppose G is a triangle-free graph on n vertices. Then $e(G) \leq \frac{n^2}{4}$

If G has n vertices and $\geq \left\lfloor \frac{n^2}{4} \right\rfloor + 1$ edges, then G has $\geq \left\lfloor \frac{n}{2} \right\rfloor$ triangles.

Theorem [Turán] If G has n vertices and is K_{r+1}-free, then $e(G) \leq \left(1 - \frac{1}{r}\right) \frac{n^2}{2}$

A **complete** r**-multipartite graph** is the analogue of the complete bipartite graph, but with r parts instead of two.

Little oh notation: $\frac{o(n)}{n}$ goes to 0 as n goes to infinity.
Let F be some forbidden graph. We denote $\mathrm{ex}(n; F) := \max\{e(G) : v(G) = n, F \nsubseteq G\}$

We can paraphrase Turán's theorem as follows:

$$\mathrm{ex}(n; K_{r+1}) = \left(1 - \frac{1}{r} + o(1)\right) \frac{n^2}{2}$$

Theorem [Erdös-Stone-Simonovits] Let F be a forbidden subgraph with $\chi(F) \geq 2$. Then

$$\mathrm{ex}(n; F) = \left(1 - \frac{1}{\chi(F) - 1} + o(1)\right) \frac{n^2}{2}$$